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SMITH THEORY FOR p-GROUPS

BY

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ABSTRACT. When a p-group G acts on a manifold, the behavior of the cohomology of the subgroups of G singles out a special collection of fixed point sets of these subgroups. A bound on the size of the spaces in this collection is derived using equivariant cohomology. For a special class of nonabelian p-groups this bound is strong enough to require that certain fixed point sets must vanish. Application of this bound to a linear representation of G yields a lower bound for the cohomology of G.

0. Introduction. The use of equivariant cohomology to generalize Smith theory was begun by A. Borel in [1]. This use of equivariant cohomology was further extended by G. Bredon in [2] and W.-Y. Hsiang in [5] and [6]. Here we derive a generalization of Smith theory which applies to all p-groups.

In [9], D. Quillen calculated the Krull dimension of the cohomology ring of finite group G. This suggests that we isolate a certain collection $\mathcal{F}_0$ of the fixed point sets of the subgroups of G when G acts on a manifold M.

When G is a p-group, repeated applications of the Thom isomorphism allow us to derive a bound on the equivariant cohomology of M. More applications of the Thom isomorphism produce a bound on the size of the fixed point sets in $\mathcal{F}_0$. When $G = Z_p$ this becomes the usual Smith theory estimate.

For certain nonabelian p-groups we obtain new information which is strong enough to cause some of the possible types of fixed point sets to vanish. For these groups we further strengthen the estimate on $\mathcal{F}_0$ by globalizing a technique used in [7].

Given a real representation r of G we have an action of G on a sphere S^m. Now we know that the fixed point sets in $\mathcal{F}_0$ are spheres and we can convert the bound on $\mathcal{F}_0$ into information about the cohomology of G. Several examples of such calculations are done.

1. Two cohomology invariants. For the purposes of this paper a manifold M will be a smooth compact manifold, with or without boundary. A submanifold F of M will be required to meet the boundary of M transversely. If F is a

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closed submanifold then the boundary of F is contained in the boundary of M.

Fix a prime number p; all cohomology will have Z_p coefficients. If A^* is a graded Z_p module, we define the Poincaré series of A^* as the formal power series

$$PS(A^*) = \sum_i (\dim_{Z_p} A^i) t^i.$$

The inequality $\Sigma_i a_i t^i \leq \Sigma_i b_i t^i$ between two such series will mean that $a_i \leq b_i$ for each i.

Let $\sigma^k A^*$ be the k-fold suspension of A^*, so $\sigma^k A^* = B^*$ where $B^m = A^{m-k}$. We have the following formulas:

(1.1) $$PS(\sigma^k A^*) = t^k PS(A^*).$$

(1.2) $$PS(A^* \oplus B^*) = PS(A^*) + PS(B^*).$$

(1.3) $$PS(A^* \otimes B^*) = PS(A^*) PS(B^*).$$

If X is a space, set $PS(X) = PS(H^*(X))$, and if G is a group, set $PS(G) = PS(H^*(G))$.

Using $PS(A^*)$ we define two invariants $l(A^*)$ and $c(A^*)$. If it makes sense, expand $PS(A^*)$ as a power series in $(1-t)$:

(1.4) $$PS(A^*) = c(A^*)(1-t)^{-l(A^*)} + \text{higher terms}$$

where $c(A^*) \neq 0$. Since the coefficients of $PS(A^*)$ are nonnegative we have that $l(A^*) \geq 0$ and $c(A^*) > 0$. It may be that $l(A^*) = \infty$ so that $c(A^*)$ is not defined, but in this paper $l(A^*)$ is always a finite nonnegative integer.

We have the following formulas:

(1.5) $$l(\sigma^k A^*) = l(A^*).$$

(1.6) $$c(\sigma^k A^*) = c(A^*).$$

(1.7) $$l(A^* \oplus B^*) = \max\{l(A^*), l(B^*)\}.$$

(1.8) $$c(A^* \oplus B^*) = \begin{cases} c(A^*) & \text{if } l(A^*) > l(B^*), \\ c(A^*) + c(B^*) & \text{if } l(A^*) = l(B^*), \\ c(B^*) & \text{if } l(A^*) < l(B^*). \end{cases}$$

(1.9) $$l(A^* \otimes B^*) = l(A^*) + l(B^*).$$

$$c(A^* \otimes B^*) = c(A^*)c(B^*). \tag{1.10}$$

Again if X is a space set $l(X) = l(H^*(X))$ and $c(X) = c(H^*(X))$, and if G is a group set $l(G) = l(H^*(G))$ and $c(G) = c(H^*(G))$. Note that if M is a manifold then $l(M) = 0$ and $c(M) = \Sigma_i \dim_{Z_p} H^i(M)$.

Our interest in these invariants arose from D. Quillen's calculation, see [9], that if G is a finite group then $l(G)$ is the p-rank of G. More precisely $l(G) = \max\{k \mid Z_p^k \subseteq G\}$ where Z_p^k is the k-fold product of Z_p with itself. When G is a p-group we obtain in [7] information about $c(G)$.

2. **Fixed point sets of subgroups.** Suppose the finite group G acts on the manifold M, we say that the element g of G moves the point x of M to the point xg. If $S \subseteq G$ and $F \subseteq M$ we set $xS = \{xh \mid h \in S\}$, $Fg = \{yg \mid y \in F\}$, and $FS = \{yh \mid y \in F \text{ and } h \in S\}$. Let $M/G = \{xG \mid x \in M\}$ be the orbit space of the action.

Define the isotropy group I_x for the point $x \in M$ as the collection of all elements $g \in G$ which fix x, so $xI_x = \{x\}$. Let $\mathcal{I} = \{I_x \mid x \in M\}$ be the collection of all isotropy groups for the action. Consider $\mathcal{I}$ as a partial order with $I_1 \leqslant I_2$ if $I_1 \subseteq I_2$.

For an isotropy group I call $F(I) = \{x \in M \mid I_x = I\}$ the total fixed point set of I. A connected component F of $F(I)$ is a fixed point set of I. Let $\mathcal{F}$ be the collection of all fixed point sets of all isotropy groups of the action. For $F \in \mathcal{F}$ let I_F be the isotropy group of F, for each $x \in F$, $I_x = I_F$.

Each member F of $\mathcal{F}$ is a submanifold of M. At a boundary point $x \in F \cap \partial M$ one can show that F meets ∂M transversely around x. In general F is not closed and its closure $\overline{F}$ is a union of fixed point sets associated with isotropy groups $J \supseteq I_F$. In fact $\mathcal{F}$ becomes a partial order by saying $F_1 \leqslant F_2$ if $F_2 \subseteq \overline{F}_1$. Now F is a closed submanifold of M precisely when F is a maximal element of the partial order $\mathcal{F}$.

Let τ be the tangent bundle of M and τ_x the fiber of τ over the point x. Now I_x acts on τ_x and this provides a real representation of I_x. If $x \in F \in \mathcal{F}$ and if $k = \dim F$ then the representation of I_x on τ_x decomposes as $k\,\mathrm{Id} + r_x$, where $k\,\mathrm{Id}$ stands for k copies of the identity representation of I_x. Since F is connected, if $y \in F$ is a second point then r_x and r_y are equivalent as real representations of $I_F = I_x = I_y$. We call $r_F = r_x = r_y$ the normal representation of the isotropy group I_F of F.

(2.1). LEMMA. *For each $g \in G$ and $F \in \mathcal{F}$ we have $Fg \in \mathcal{F}$ and $I_{(Fg)} = g^{-1}(I_F)g$.*

Thus G acts on $\mathcal{F}$. Define the normalizer N_F of F as the collection of all $g \in G$ with $Fg = F$; then N_F is a subgroup of $N_G(I_F) = \{g \in G \mid g^{-1}(I_F)g = I_F\}$, the normalizer of I_F in G. So I_F is a normal subgroup of N_F and we define the Weyl group W_F of F as the quotient N_F/I_F.

(2.2). LEMMA. *For $F \in \mathcal{F}$ the group W_F acts freely on F.*

PROOF. An element of W_F is a coset $(I_F)u$ of I_F in N_F. Define the action of W_F on F by having $(I_F)u$ move a point $x \in F$ to the point $x(I_F)u = (xI_F)u = xu \in F$. If $xu = x$ then $u \in I_F$ and $(I_F)u = I_F$ is the identity in W_F, and the action is free.

If r is a representation of a subgroup I of G and if g is an element of G then define the conjugate representation r^g of the subgroup $g^{-1}Ig$ by $r^g(g^{-1}hg) = r(h)$ for $h \in I$. Define the normalizer $N_G(r)$ of r as the collection of $g \in N_G(I)$ for which r^g is equivalent to r as representations of I.

(2.3). LEMMA. *For each $g \in G$ and $F \in \mathcal{F}$ we have $r_{(Fg)} = (r_F)^g$.*

Thus N_F is a subgroup of $N_G(r_F)$. A more detailed discussion of the above material can be found in [8]. For the purposes of extending Smith theory we single out a subset $\mathcal{F}_0$ of $\mathcal{F}$. This subset consists of all $F \in \mathcal{F}$ such that

(a) F is maximal in the partial order $\mathcal{F}$.

(b) $l(I_F) = l(G)$.

(c) If $E < F$ in $\mathcal{F}$ then $l(I_E) < l(G)$.

The third condition can be stated in terms of the normal representation r_F. This requires a discussion of linear actions.

Suppose r is a representation of the subgroup I of G; then r is equivalent to an orthogonal representation and induces an action of I on some disk D^m. If J is a subgroup of I let $D(J)$ be the collection of all points $x \in D^m$ left invariant by J, that is $xJ = \{x\}$; then $D(J)$ is a linear subdisk of D^m, that is, the intersection of a linear subspace with D^m. The total fixed point set for J is given by

$$F(J) = D(J) - \bigcup\{D(L) \mid J \subset L \subseteq I\}. \tag{2.4}$$

Let $\mathcal{I}(r)$ consist of all $J \subseteq I$ with $F(J) \neq \emptyset$. Since the origin 0 is fixed by I we have $0 \in F(I)$ and $I \in \mathcal{I}(r)$.

Return to the original situation where G acts on M.

(2.5). LEMMA. *The element $F \in \mathcal{F}$ is in $\mathcal{F}_0$ if and only if*

(a) *F is maximal in $\mathcal{F}$.*

(b) *$l(I_F) = l(G)$.*

(c′) *If $J \in \mathcal{I}(r_F) - I_F$ then $l(J) < l(G)$.*

PROOF. We need only show that conditions (c) and (c′) are equivalent. If x is a point of F then for some neighborhood U_x of x in M the action of I_x on U_x is equivariantly diffeomorphic to the action of I_x on τ_x. This action decomposes as $k\,\mathrm{Id} + r_F$.

Now if $E < F$ in the partial order $\mathfrak{F}$ then $F \subset \bar{E}$ and so for some $x \in F$ we have $x \in \bar{E}$ and E meets U_x. The isotropy group I_E then occurs in $\mathfrak{I}(r_F)$.

Conversely each $J \in \mathfrak{I}(r_F) - I_F$ occurs as I_y for some $y \in U_x$. By (2.4) we can connect y to x by a path $\gamma(t)$ so that $\gamma(1) = y$, $\gamma(0) = x$, and $I_{\gamma(t)} = J$ if $0 < t \leqslant 1$. Thus $\gamma((0, 1])$ must lie in some component E of $F(J)$ and $J = I_E$ while $x \in \bar{E}$ or $E < F$ in $\mathfrak{F}$.

3. **An estimate for equivariant cohomology.** After defining equivariant cohomology $H_G^*(M)$ for an action of G on M, we use the Thom isomorphism and equivariant tubular neighborhoods to obtain an estimate for the invariant $l(H_G^*(M))$ in terms of the isotropy groups of the action.

By [11, §19], there is an N-connected manifold E_G^N on which the finite group G acts freely. For $k < N$ we define $H_G^k(M) = H^k(M \times_G E_G^N)$. Here if Y and Z are any two spaces on which G acts, then $Y \times_G Z$ is the orbit space $(Y \times Z)/G$ of the diagonal action of G on the product space $Y \times Z$; $g \in G$ moves the point (y, z) to the point (yg, zg).

(3.1). THEOREM (SERRE). *The following estimate holds for equivariant cohomology*:

$$PS(H_G^*(M)) \leqslant PS(G)PS(M).$$

PROOF. Since G acts freely on E_G^N with orbit space B_G^N, a classifying space for G up through dimension N, we can view $M \times_G E_G^N$ as a bundle over B_G^N with fiber M. By [10] there is a spectral sequence with $E_2^{**} = H^*(B_G^N; H^*(M))$ which converges to $H^*(M \times_G E_G^N)$. Thus

$$\begin{aligned} PS(H^*(M \times_G E_G^N)) &\leqslant PS(H^*(B_G^N; H^*(M))) \\ &\leqslant PS(H^*(B_G^N) \otimes H(M)) \\ &= PS(B_G^N)PS(M). \end{aligned}$$

Letting N approach infinity we obtain the theorem as stated.

Next we obtain an estimate on $H_G^*(M)$ which allows us to see the effect of fixed point sets.

(3.2). LEMMA. *Suppose a p-group G acts on a manifold M with F a closed invariant submanifold. If p is odd, then assume that F has an oriented normal bundle V_F. Let $t(F)$ be an open, invariant, tubular neighborhood for F; then*

$$PS(H_G^*(M)) \leqslant t^k PS(H_G^*(F)) + PS(H_G^*(M - t(F)))$$

where $k = \dim M - \dim F$.

PROOF. By [3, §22], we know that $t(F)$ exists. If p is odd then, since G has odd order, the action of G on ν_F preserves orientation. So the submanifold $F \times_G E_G^N$ has an oriented normal bundle in $M \times_G E_G^N$. Also $t(F) \times_G E_G^N$ serves as a tubular neighborhood for $F \times_G E_G^N$. Applying the Thom isomorphism, see [12], we have

$$H^*(M \times_G E_G^N, (M - t(F)) \times_G E_G^N) = \sigma^k H^*(F \times_G E_G^N).$$

Letting N become large this gives

$$H_G^*(M, M - t(F)) = \sigma^k H_G^*(F).$$

When $p = 2$ we are using Z_2 coefficients and the Thom isomorphism holds with no assumptions on ν_F.

From the exact triangle:

$$\begin{array}{ccc} & \sigma^k H_G^*(F) & \\ \nearrow & & \searrow \\ H_G^*(M - t(F)) & \longleftarrow & H_G^*(M) \end{array}$$

we obtain the estimate:

$$\dim_{Z_p} H_G^m(M) \leqslant \dim_{Z_p} H_G^{m-k}(F) + \dim_{Z_p} H_G^m(M - t(F)).$$

This is equivalent to the stated inequality on Poincaré series.

To apply (3.2) properly we need the following information about normal bundles of fixed point sets.

(3.3). LEMMA. *If a p-group G acts on a manifold M with p odd and if F is any connected component of the total fixed point set of a subgroup I of G, then ν_F is oriented.*

PROOF. Proceed by induction on the order of I. If I is the trivial subgroup then $\dim F = \dim M$, and ν_F has 0 dimensional fibers and is oriented. Otherwise since I is a p-group it possesses a central subgroup K of order p, see [4, Chapter 4].

Consider the action of K on M, since K fixes each point of F there is a connected component E of the total fixed point set of K which contains F. By [3, §38], we know that ν_E, the normal bundle of E in M, is oriented. Now $\nu_F = \nu_E + \nu_F^E$ where ν_F^E is the normal bundle of F as a submanifold of E.

The quotient group I/K acts on E. To see this suppose $g \in I$, $h \in K$, and $x \in E$; then $(xg)h = (xh)g = xg$ and xg is a fixed point of K. Thus Eg is a connected component of the total fixed point set of K. Since $F \subseteq E \cap Eg$ we must have $E = Eg$. Thus I acts on E and since K fixes E pointwise this induces an action of I/K.

The set F is a connected component of the fixed point set of I/K as it acts on E. By induction V_F^E is oriented and thus V_E, a sum of oriented bundles, is oriented.

(3.4). THEOREM. *If a p-group G acts on a manifold M then*

$$l(H_G^*(M)) \leq \max \{l(I) | I \in \mathcal{I}\}$$

where $\mathcal{I}$ is the collection of isotropy groups for this action.

PROOF. The collection $\mathcal{F}$ of connected components of fixed point sets of subgroups is finite. To see this note that $\mathcal{I}$ is finite since G is finite. For a given subgroup I, the total fixed point set $F(I)$ has only finitely many connected components. Otherwise by the compactness of M there would be a point x with infinitely many such components in its neighborhood. But x has a neighborhood U_x such that $U_x \cap U_{xg} = \emptyset$ unless $g \in I_x$. Further I_x acts on U_x in a way equivariantly diffeomorphic to the action of I_x on τ_x. Formula (2.4) shows that a linear action has only a finite number of connected components of fixed point sets of subgroups.

Proceed by induction on the size of $\mathcal{F}$. If F is a maximal element in the partial order $\mathcal{F}$ then FG is an invariant, closed, submanifold of M. If p is odd then by (3.3) V_{FG} is oriented. So (3.2) gives

$$PS(H_G^*(M)) \leq t^k PS(H_G^*(FG)) + PS(H_G^*(M - t(FG))). \tag{3.5}$$

Using the definition of N_F and W_F from §2 we have that

$$(FG) \times_G E_G^N = F \times_{N_F} E_G^N = F \times_{W_F} B_{I_F}^N$$

where $B_{I_F}^N = E_G^N / I_F$ is a classifying space for I_F through dimension N. By (2.2) W_F acts freely on F and $F \times_{W_F} B_{I_F}^N$ may be viewed as a bundle over F/W_F with fiber $B_{I_F}^N$. As in the proof of (3.1) this gives the estimate

$$PS(H_G^*(FG)) \leq PS(F/W_F) PS(I_F).$$

Since F/W_F is a manifold, $l(F/W_F) = 0$ and we have

$$l(H_G^*(FG)) \leq l(I_F). \tag{3.6}$$

Next the space $M - t(FG)$ has a corner where $t(FG)$ meets the boundary of M. After straightening the corner, $M - t(FG)$ becomes a manifold on which G acts. Further the collection of connected components of fixed point sets of subgroups for $M - t(FG)$ may be considered as a proper subset of $\mathcal{F}$. By induction we have

$$l(H_G^*(M - t(FG))) \leq \max \{l(I) \mid I \in \mathcal{I}\}. \tag{3.7}$$

Combining (3.5), (3.6), and (3.7) proves the theorem.

4. Smith theory for p-groups. By exploiting the invariant $c(G)$ we obtain an inequality involving the collection $\mathcal{F}_0$ of fixed point sets of subgroups. When $G = Z_p$ this becomes equivalent to classical Smith theory. Any new information we obtain occurs when G is nonabelian.

Suppose a p-group G acts on a manifold M with F an invariant submanifold. We say that F is isolated if for each $x \in F$ there is some neighborhood U_x of x such that for $y \in U_x - F$ we have $l(I_y) < l(H_G^*(F))$.

(4.1). PROPOSITION. *Suppose a p-group G acts on a manifold M with $F_1, F_2, \ldots, F_m$ closed, invariant, disjoint, and isolated submanifolds. If p is odd then suppose that each V_{F_i} is oriented. If $l(H_G^*(F_i)) \geq l(H_G^*(M))$ for each i then $l(H_G^*(F_i)) = l(H_G^*(M))$ and $\sum_{i=1}^m c(H_G^*(F_i)) \leq c(H_G^*(M))$.*

PROOF. Choose disjoint, open, invariant, tubular neighborhoods $t(F_1)$, $t(F_2), \ldots, t(F_m)$. Let $\partial t(F_i)$ be the boundary of $t(F_i)$. Since F_i is compact and isolated we can choose $t(F_i)$ small enough so that $l(I_y) < l(H_G^*(F_i))$ for $y \in \partial t(F_i)$. By (3.4) we have $l(H_G^*(\partial t(F_i))) < l(H_G^*(F_i))$.

As in the proof of (3.2), we obtain from the Thom isomorphism

$$H_G^*(M, M - t(F_i)) = \sigma^{k_i} H_G^*(F_i)$$

where $k_i = \dim M - \dim F_i$. Let $t(F) = \bigcup_{i=1}^m t(F_i)$; then the pair $(M, M - t(F))$ generates in equivariant cohomology the exact triangle:

$$\begin{array}{ccc} & \sum_{i=1}^{m} \sigma^{k_i} H_G^*(F_i) & \\ {\scriptstyle \delta^*} \nearrow & & \searrow {\scriptstyle j^*} \\ H_G^*(M - t(F)) & \longleftarrow & H_G^*(M) \end{array}$$

Since δ^* factors through $H_G^*(\partial t(F)) = \sum_{i=1}^m H_G^*(\partial t(F_i))$ we have

$$l(\text{image}(\delta^*) \cap \sigma^{k_i} H_G^*(F_i)) < l(H_G^*(F_i)).$$

So for the purposes of calculating the invariant c we may view j^* as an inclusion and we have that $l(H_G^*(M)) \geq l(H_G^*(F_i))$. Thus $l(H_G^*(M)) = l(H_G^*(F_i))$ and we have that

$$\sum_{i=1}^{m} c(H_G^*(F_i)) \leq c(H_G^*(M)).$$

Under stronger assumptions this becomes an equality.

(4.2). PROPOSITION. *With the same assumptions as in* (4.1), *if* $l(I_x) < l(H_G^*(M))$ *for all* $x \in M - \bigcup_{i=1}^{m} F_i$ *then*

$$\sum_{i=1}^{m} c(H_G^*(F_i)) = c(H_G^*(M)).$$

PROOF. With the new assumption we have $l(H_G^*(M - t(F))) < l(H_G^*(M))$ by (3.4) and j^* may be viewed as an isomorphism for the purposes of evaluating the invariant c.

In order to apply (4.1) we need two results from [7]. If A^* is a graded Z_p module on which G acts, let $\{A^*\}^G$ be the graded Z_p module of elements left fixed by G.

(4.3). THEOREM. *If a finite cyclic group W acts freely on a manifold M then*

$$PS(\{H^*(M)\}^W) \leq PS(M/W) \leq (1 - t)^{-1} PS(\{H^*(M)\}^W).$$

(4.4). LEMMA. *If the cyclic p-group W acts on the graded* Z_p *module* A^* *via linear transformations, then*

$$PS(A^*) \leq |W| PS(\{A^*\}^W).$$

(4.5). PROPOSITION. *If a p-group G acts freely on a manifold M then* $PS(M) \leq |G| PS(M/G)$.

PROOF. Proceed by induction on the order of G. Let K be a normal subgroup of G with the quotient $W = G/K$ a cyclic group, see [4, Chapter 4]. Since G acts freely on M, W acts freely on M/K. Applying (4.4) and (4.3) we have

$$\begin{aligned} PS(M/K) &\leq |W| PS(\{H^*(M/K)\}^W) \\ &\leq |W| PS(M/G). \end{aligned}$$

By induction $PS(M) \leq |K| PS(M/K) \leq |G| PS(M/G)$.

Finally we turn to the special collection $\mathcal{F}_0$ of fixed point sets of subgroups. Now G acts on $\mathcal{F}_0$, let $\mathcal{F}'_0 = \mathcal{F}_0/G$ be the orbit spaces. Choose a basepoint F for each orbit and label the orbit F'.

(4.6). THEOREM. *Suppose a p-group G acts on a manifold M, then*

$$\sum_{F' \in \mathcal{F}'_0} c(I_F) c(F)/|W_F| \leq c(G) c(M).$$

PROOF. For each $F' \in \mathfrak{F}'_0$ form the invariant submanifold FG. As F' runs over $\mathfrak{F}'_0$ the sets FG run over a collection of closed, invariant, disjoint submanifolds of M.

Each FG is isolated, this follows from $F \in \mathfrak{F}_0$ once we show that $l(H^*_G(FG)) = l(G)$. As in the proof of (3.4) we have that $H^k_G(FG) = H^k(F \times_{W_F} B^N_{I_F})$ for $k < N$. By (4.5) this gives

$$(4.7) \qquad PS(H^*_G(FG)) \geqslant PS(F \times B_{I_F})/|W_F| = PS(F)PS(I_F)/|W_F|.$$

As in the proof of (3.4) we have that

$$PS(H^*_G(FG)) \leqslant PS(F/W_F)PS(I_F).$$

Since F and F/W_F are manifolds, this gives $l(H^*_G(FG)) = l(I_F) = l(G)$.

Next by (3.1) we have $l(H^*_G(M)) \leqslant l(G) = l(H^*_G(FG))$ for each $F' \in \mathfrak{F}'_0$. By (3.3) if p is odd then V_{FG} is oriented and (4.1) applies. We conclude that $l(G) = l(H^*_G(M))$ and that $\Sigma_{F' \in \mathfrak{F}'_0} c(H^*_G(FG)) \leqslant c(H^*_G(M))$. By (3.1) we have $c(H^*_G(M)) \leqslant c(G)c(M)$. By (4.7) we have $c(H^*_G(FG)) \geqslant c(I_F)c(F)/|W_F|$. Combining the last three inequalities we obtain the theorem as stated.

When G is an elementary abelian p-group Z^k_p then $\mathfrak{F}_0$ is the collection of connected components of the total fixed point set $F(G)$ of G. So for each $F \in \mathfrak{F}_0$ we have $I_F = G$ and W_F is the trivial group. So (4.6) reduces to $c(F(G)) \leqslant c(M)$, which is classical Smith theory.

If G acts on a finite simplicial complex K with each $g \in G$ moving each simplex of K to another simplex of K in an affine manner, then we can imbed K in a manifold M on which G acts such that $K \subset M$ is an equivariant deformation retract. Applying (4.6) to M we recover information about K.

5. **A special sequence of groups.** The result from the previous section is applied to certain p-groups whose structure makes this estimate of particular interest. We find a condition which forces certain fixed point sets to vanish. For these groups we derive an improved estimate for equivariant cohomology which in turn strengthens the restrictions on the fixed point sets.

Let L_m be the p-group given by the presentation

$$L_m = \langle a, b \,|\, a^{p^m} = b^p = 1, ab = ba^{1+p^{m-1}} \rangle.$$

If $p = 2$ we require that the integer m be at least 3, otherwise it must be at least 2. In [7] we established

(5.1). LEMMA. *For the group L_m, the invariant $c(L_m) = 1/p$.*

Consider the subgroup I_m of L_m generated by the elements a^p and b; then I_m is isomorphic to $Z_{p^{m-1}} \times Z_p$. Suppose L_m acts on a manifold M;

we wish to consider elements $F \in \mathcal{F}_0$ with $I_F = I_m$. Notice that $l(I_m) = l(L_m) = 2$ so this is possible. We are interested in the case when (4.6) produces the greatest restriction on F, namely we want W_F to be the trivial group. Since W_F is a subgroup of $N_G(r_F)$, we have that W_F is trivial if $(r_F)^a \neq r_F$. When this occurs r_F cannot be the restriction of a representation on all of L_m and F must be maximal in $\mathcal{F}$.

Let A_m be the subgroup generated by $a^{p^{m-1}}$ and b. Condition (c′) of (2.5) is satisfied precisely when r_F restricted to A_m does not contain any copies of the identity representation of A_m.

The collection $\mathcal{R}_0$ of representations which satisfy the above conditions can be made explicit as follows. Let α_k be the representation of $Z_{p^{m-1}}$ given by $\alpha_k(a^p) = e^{(2\pi i k)/p^{m-1}}$, here k is to be considered modulo p^{m-1}. Let β_l be the representation of Z_p given by $\beta_l(b) = e^{(2\pi i l)/p}$, here l is to be considered modulo p. The general irreducible complex representation of I_m is $\alpha_k \times \beta_l$. The action induced by conjugation by a is given by $(\alpha_k \times \beta_l)^a = \alpha_k \times \beta_{(k+l)}$.

If r is a real representation of I_m, consider it as a complex representation and decompose it as a sum $r = \Sigma_{k,l} m(k,l)\alpha_k \times \beta_l$. Now r is real precisely when $m(k, l) = m(-k, -l)$. Since r restricted to A_m does not contain the identity representation we must have that if $m(k, l) \neq 0$ then either $(k, p) = 1$ or $(l, p) = 1$. Since $r^a \neq r$ we must have that, for some k, l, $m(k, k+l) \neq m(k, l)$. Let $\mathcal{R}_0$ be the collection of all real representations of I_m which satisfy these conditions.

(5.2). THEOREM. *Suppose L_m acts on a manifold M with $c(M) < p$ then there are no fixed point sets F of the subgroup I_m which have a normal representation $r_F \in \mathcal{R}_0$.*

PROOF. By our definition of $\mathcal{R}_0$ we know that $r_F \in \mathcal{R}_0$ means that $F \in \mathcal{F}_0$. By (4.6) we know that

$$c(F)c(I_m)/|W_F| \leqslant c(M)c(L_m).$$

Since $r_F \in \mathcal{R}_0$ we have $|W_F| = 1$. Since I_m is abelian, $c(I_m) = 1$ and by (5.1) $c(L_m) = 1/p$. So we have $c(F) \leqslant c(M)/p < 1$ and F must be empty.

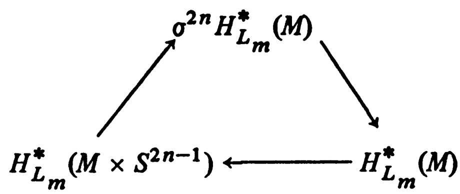
This theorem places restrictions on the actions of other p-groups. If a p-group G acts on a manifold M and K is a subgroup, then let $E(K) = \{x \in M \mid xK = \{x\}\}$. Let F be any connected component of $E(K)$ and set $N_F = \{g \in N_G(K) \mid Fg = F\}$; then the quotient $W_F = N_F/K$ acts on F but not necessarily freely. If W_F contains L_m then L_m acts on F. By ordinary Smith theory $c(E(K)) \leqslant c(M)$ and if $c(M) < p$ then $c(F) < p$ and (5.2) places restrictions on the fixed points of I_m. This in turn translates into restrictions on the original action.

By globalizing the technique used in [7] to calculate $c(L_m)$ we can improve the estimate (3.1) for $H^*_{L_m}(M)$.

(5.3). PROPOSITION. *If L_m acts on a manifold M and if $l(H^*_{L_m}(M)) = 2$ then $c(H^*_{L_m}(M)) \leq c(\{H^*(M)\}^B)/p$ where B is the subgroup of L_m generated by b.*

PROOF. The group L_m has an irreducible complex representation r such that when restricted to the subgroup generated by $a^{p^{m-1}}$ it does not contain the identity representation. If $n = \dim_C r$ then r produces an action of L_m on D^{2n}. The only isotropy groups that occur are L_m, conjugates of B, and the trivial group. Now L_m fixes only the origin while B fixes a subdisk $D^{2k} \subset D^{2n}$. Since B has p conjugates, $kp = n$.

Consider the diagonal action of L_m on $M \times D^{2n}$, since D^{2n} is contractible we have $H^*_{L_m}(M) = H^*_{L_m}(M \times D^{2n})$. Using the Thom isomorphism, the pair $(M \times D^{2n}, M \times S^{2n-1})$ gives an exact triangle:



This gives an estimate:

$$PS(H^*_{L_m}(M)) \leq (1 - t^{2n})^{-1} PS(H^*_{L_m}(M \times S^{2n-1}))$$

which tells us that:

$$c(H^*_{L_m}(M)) \leq (1/2n) c(H^*_{L_m}(M \times S^{2n-1})).$$

If $S^{2k-1} \subset S^{2n-1}$ is the set of points fixed by B then (4.2) applies to the invariant submanifold $M \times (S^{2k-1} L_m)$ and we find that

$$c(H^*_{L_m}(M \times S^{2n-1})) = c(H^*_{L_m}(M \times (S^{2k-1} L_m))).$$

Let $K = N_{L_m}(B)$ and $W = K/B$; then we have the following identification:

$$\begin{aligned} H^*_{L_m}(M \times (S^{2k-1} L_m)) &= H^*_K(M \times S^{2k-1}) \\ &= H^*(E_K \times_K (M \times S^{2k-1})) \\ &= H^*((E_K \times M) \times_K S^{2k-1}) \\ &= H^*((E_K \times_B M) \times_W S^{2k-1}). \end{aligned}$$

Now W acts freely on S^{2k-1} and we have the estimate:

$$PS((E_K \times_B M) \times_W S^{2k-1}) \leqslant PS(E_K \times_B M)PS(S^{2k-1}/W)$$
$$\leqslant (1 + t + t^2 + \cdots + t^{2k-1})(1 - t)^{-1}PS(\{H^*(M)\}^B)$$

where the last estimate uses (4.3) and the fact that S^{2k-1}/W is a lens space.

Collecting estimates we find that

$$c(H^*_{L_m}(M)) \leqslant (2k/2n)c(\{H^*(M)\}^B)$$

which is the asserted estimate since $kp = n$.

6. **Application to group cohomology.** By applying (4.6) to a situation where the fixed point sets are known we obtain a lower bound for $c(G)$. Combined with an upper bound from [7], these inequalities determine $c(G)$ for some groups, several examples are given.

Suppose r is a real representation of the p-group G, this gives an action of G on S^n. We require that r does not contain any copies of the identity representation so that G does not fix any point of S^n and $G \notin \mathcal{I}$, the collection of isotropy groups. By (2.4) if $I \in \mathcal{I}$ is a maximal element of the partial order then $F(I)$ is an m-sphere.

Since r is a representation of G we have $r^g = r$ for $g \in G$. If $g \in N_G(I_x)$ then this means that $(r_x)^g = r_x$. This leads to the identification $W_F = N_G(I_F)/I_F$ for $F \in \mathcal{F}$.

If we define $\mathcal{I}_0$ as the collection of all $I \in \mathcal{I}$ such that I is maximal in $\mathcal{I}$, $l(I) = l(G)$, and if $J \in \mathcal{I}$ with $J \subset I$ then $l(J) < l(G)$, then (4.6) rephrases as

(6.1). THEOREM. *For any representation r of a p-group G which does not contain the identity representation the following estimate holds*:

$$\sum_{I' \in \mathcal{I}'_0} |I'|c(I')/|N_G(I')| \leqslant c(G).$$

The following special case contains no mention of representations.

(6.2). THEOREM. *Suppose a p-group G contains normal subgroups $N_1, N_2, \ldots, N_k$ such that*

(a) *each quotient G/N_i is cyclic,*

(b) *for each i, $l(N_i) = l(G)$,*

(c) *for each $i \neq j$, $l(N_i \cap N_j) < l(G)$,*

then $\Sigma_{i=1}^k |N_i|c(N_i)/|G| \leqslant c(G)$.

PROOF. Since G/N_i is cyclic there is an irreducible representation r_i of G whose kernel is N_i. Apply (6.1) to $r = r_1 + r_2 + \cdots + r_k$.

To illustrate how (6.2) may lead to the identification of $c(G)$ we recall a result from [7].

(6.3). THEOREM. *If G is a p-group whose center $Z(G)$ is cyclic while the* *quotient $G/Z(G)$ is isomorphic to Z_p^k for some $k > 0$ and if, when $p = 2$, G has no subquotient group isomorphic to the quaternion group, then $c(G) \leqslant e(G)/p$. Here $e(G)$ is the number of subgroups of G isomorphic to $Z_p^{l(G)}$.*

Consider the groups L_m from §5. By (6.3) we calculate that $c(G) \leqslant 1/p$, while by (6.2) we have $c(G) \geqslant 1/p$.

As a second example restrict attention to p odd and consider the group G of order p^3 given by the presentation

$$\langle a, b, z | a^p = b^p = z^p = 1, az = za, bz = zb, ab = baz \rangle.$$

Again (6.3) applies and we calculate that $c(G) \leqslant (p + 1)/p$. If $N_1, N_2, \ldots, N_{p+1}$ are the maximal subgroups of G then (6.2) applies and we obtain $c(G) \geqslant (p + 1)/p$.

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